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NOTE

ON STABILIZERS OF SOME MOIETIES OF THE RANDOM TOURNAMENT

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Any countably infinite tournament T_0 embeds as a moiety of the random tournament T in such a way that its setwise stabilizer in $\mathrm{Aut}(T)$ is isomorphic to $\mathrm{Aut}(T_0)$.

1. Result and consequence

There is an interesting problem on permutation groups, and more specifically on groups of automorphisms of universal structures, which apparently has not been considered systematically. This can be stated more or less formally as follows in the framework of model theory. Assume that $\mathcal{C}$ is a class of structures in a fixed language, with a universal model $\mathfrak{M}$, and let $\mathcal{G}$ denote the class of groups of automorphisms of structures in $\mathcal{C}$. Then is it true that the group of automorphisms $\mathrm{Aut}(\mathfrak{M})$ is universal for the class of groups $\mathcal{G}$? And, if the answer is positive, in what sense is it universal?

In this note we give a positive answer when $\mathcal{C}$ is the class of countable tournaments. In this case, the universal model $\mathfrak{M}$ is the (countable) random tournament. It is the analog for tournaments of the countable universal graph in the class of graphs, also known as the random graph. With tournaments, the class of groups $\mathcal{G}$ typically consists of all groups without involutions.

Our main result is the following.

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Theorem 1.1. *Let T_0 be a countably infinite tournament. Then T_0 embeds into the random tournament T, as a moiety of T, in such a way that the restriction to T_0 of automorphisms of $\mathrm{Aut}(T)_{\{T_0\}}$ is an isomorphism from $\mathrm{Aut}(T)_{\{T_0\}}$ onto $\mathrm{Aut}(T_0)$ (equivalently, in such a way that each automorphism of T_0 extends to a unique automorphism of T).*

A *tournament* is a set of elements, called *vertices*, equipped with a binary relation which is irreflexive, complete, and antisymmetric. In other words, it is a complete oriented graph. If X and Y are two disjoint subsets of vertices of a tournament, we say that X *dominates* Y if (y,x) is in the relation for every $(x,y) \in X \times Y$. The *random* tournament T is the countably infinite tournament determined by the following axioms: for every pair of finite disjoint subsets of vertices A and B, there is a vertex z dominating A and dominated by B. A classical back-and-forth argument shows that the first-order theory of such a tournament is ω-categorical, or equivalently that T is uniquely determined up to isomorphism by these axioms. Obviously, these axioms also imply that T is universal for countable tournaments.

Some notation and terminology of Theorem 1.1 remains to be defined. If Δ is a subset of a tournament T and G denotes $\mathrm{Aut}(T)$, then we use the standard notation $G_{\{\Delta\}}$ from permutation group theory to denote the setwise stabilizer of Δ in G. Following P. Neumann, we call *moiety* any infinite coinfinite subset of a countably infinite set. Hence, Theorem 1.1 gives the universality of $\mathrm{Aut}(\mathfrak{M})$, with $\mathfrak{M}$ the random tournament, for the class of groups of automorphisms of countable tournaments, more specifically as setwise stabilizers of moieties.

We would of course prefer to have another characterisation of these groups. Obviously they are without involutions, as an automorphism of a tournament cannot swap two distinct vertices. A converse was first shown in the finite case.

Fact 1.2 ([5])**.** *Let G be a finite group without involutions. Then there is a finite tournament T such that G is isomorphic to $\mathrm{Aut}(T)$.*

Note that, for finite groups without involutions different from ${\mathbb{Z}_3}^2$ and ${\mathbb{Z}_3}^3$, the group can even be regular as it is shown in [2] or more correctly in [3]. In the infinite case the following was shown later.

Fact 1.3 ([1])**.** *Let G be an infinite group without involutions. Then there is a tournament T such that $\mathrm{Aut}(T) \simeq G$ and $\mathrm{Aut}(T)$ has five orbits on the vertex set of T, two of which consist of a single vertex each.*

Possible subgroups of the group of automorphisms of the random tournament have already been studied in [4]. It is shown there that any countably infinite group without involutions embeds as a regular subgroup of the group of automorphisms of the random tournament. Theorem 1.1 about stabilizers of moieties takes up the universality problem from another end and has the following consequence.

Corollary 1.4. *Let G be a countable (finite or infinite) group without involutions. Then G embeds into the automorphism group of the random tournament as the setwise stabilizer of a moiety.*

Proof. By Theorem 1.1, it suffices to show that G is the automorphism group of a countably infinite tournament. This is given by Fact 1.3 if G is infinite. If G is finite, replace the finite tournament T given by Fact 1.2 by the infinite tournament $\mathbb{N} \sqcup T$ (disjoint union), where $\mathbb{N}$ is equipped with its natural order and where T dominates $\mathbb{N}$. ∎

A priori one may wonder whether the preceding corollary holds for any group without involutions and of cardinality at most $2^{\aleph_0}$. The answer to this question is negative, as it is known from [7] that an uncountable free group cannot be the group of automorphisms of a countable structure. In this sense, Theorem 1.1 is optimal with regards to the cardinalities of the groups involved.

Our proof of Theorem 1.1 is a mere elaboration on Lemma 2.1 of [6], whose roots can be tracked in [8, pages 244–245]. That lemma concerned graphs rather than tournaments. Everything mentionned here for tournaments could be shown for graphs as well, with the class of all groups replacing the class of groups without involutions. We prefer tournaments as they can be much more "rigid" than graphs for these kinds of questions.

To deal with tournaments, we will eventually need the following result. Recall that a group G acts *without interchanges* on a set Ω if no element of G swaps two distinct elements of Ω.

Fact 1.5 ([1]). *Given a faithful permutation group (P, Ω) without interchanges, there is a tournament T such that $\Omega \subseteq T$, Ω is setwise stabilized by $\mathrm{Aut}(T)$, and the restriction of automorphisms of T to Ω is an isomorphism from $\mathrm{Aut}(T)$ onto P (in particular $\mathrm{Aut}(T)|\Omega = P$).*

2. Proof of Theorem 1.1

The following well-known lemma generalizes the fact that an automorphism of a tournament cannot have orbits with two elements.

Lemma 2.1. *An automorphism of a tournament cannot swap two finite distinct subsets of vertices.*

Proof. First we remark that an automorphism σ of a tournament cannot have cycles of even cardinality. For if x, $\sigma(x)$, $\sigma^2(x)$, $\dots$, $\sigma^{2k}(x) = x$ were such a cycle of cardinality $2k$ for some $k \geq 1$, then σ^k would swap x and $\sigma^k(x)$, which is impossible.

Suppose now that X and Y are two finite distinct subsets of vertices swapped by an automorphism σ of a tournament, that is $\sigma(X) = Y$ and $\sigma(Y) = X$. Then we clearly have that $X \cup Y$ is a union of finite cycles of σ. Now if z, $\sigma(z)$, $\dots$, $\sigma^{2k+1}(z) = z$ is such a cycle of cardinality $2k+1$, then the fact that $\sigma(X) = Y$ and $\sigma(Y) = X$ implies that the cycle is in $X \cap Y$. Thus $X \cup Y \subseteq X \cap Y$ and $X = Y$, which is contrary to our assumption. ∎

We shall now embark on the proof of Theorem 1.1. Let $(T_i)_{i \geq 1}$ be a sequence of countably infinite and pairwise disjoint sets, all disjoint from T_0. Let T be the disjoint union of all the T_i's ($i \geq 0$), T_0 carrying its given tournament structure. We will complete T to make it into the random tournament. For $i \geq 1$, let $\mathfrak{S}_i$ denote the set of (finite) subsets S of $T_0 \sqcup \cdots \sqcup T_{i-1}$ with i^2 elements, such that S contains exactly i elements of T_s ($0 \leq s \leq i-1$). Fix also $(\psi_i)_{i \geq 1}$, a sequence of bijections from $\mathfrak{S}_i$ onto T_i.

By induction on $i \geq 0$, we build an increasing sequence of supertournaments of T_0 on $T_0 \sqcup \cdots \sqcup T_i$ with the following property:

(1) *For every $s \in \{1, \dots, i\}$ and every $S \in \mathfrak{S}_s$, $\psi_s(S)$ is dominated by S and dominates every vertex of $(T_0 \sqcup \cdots \sqcup T_{s-1}) \setminus S$.*

We pass from i to $i+1$ as follows. Assume that the tournament on $T_0 \sqcup \cdots \sqcup T_i$ is built. Then the group

$$G = \mathrm{Aut}(T_0 \sqcup \cdots \sqcup T_i)_{\{T_0\}}$$

stabilizes each T_s ($1 \leq s \leq i$), as each such T_s is the set of vertices dominated by exactly s vertices of T_0. Thus the action of G on $T_0 \sqcup \cdots \sqcup T_i$ induces naturally an action on $\mathfrak{S}_{i+1}$.

Claim 2.2. *The action of G on $\mathfrak{S}_{i+1}$ is faithful.*

Proof. Assume $g \in G$ acts trivially on $\mathfrak{S}_{i+1}$. We have to show that g stabilizes $T_0 \sqcup \cdots \sqcup T_i$ pointwise. So let x be a vertex of this tournament. Then $x \in T_s$ for some $s \in \{0, \dots, i\}$. Now one can pick two subsets U and V of T_s containing exactly $i+1$ vertices and such that $U \cap V = \{x\}$. Then $g(x) = g(U \cap V) = g(U) \cap g(V) = U \cap V = x$. Our claim is proved. ∎

By Claim 2.2 and Lemma 2.1, G acts faithfully and without interchanges on $\mathfrak{S}_{i+1}$. Fact 1.5 now shows that there is a tournament structure on T_{i+1}

(identified with $\mathfrak{S}_{i+1}$ via ψ_{i+1}) compatible with the action of G. We now consider the supertournament structure of $T_0 \sqcup \cdots \sqcup T_i$ on $T_0 \sqcup \cdots \sqcup T_{i+1}$ as follows: T_{i+1} carries the tournament structure as above, and for every $x \in T_{i+1}$, x is dominated by $\psi_{i+1}^{-1}(x)$ and dominates $(T_0 \sqcup \cdots \sqcup T_i) \setminus \psi_{i+1}^{-1}(x)$. Our requirement (1) is met at step $i+1$ and this ends our construction at step $i+1$.

Now it is clear that the tournament on $T = \bigsqcup_{i=0}^{\infty} T_i$ satisfies the axioms of the random tournament, so it is the random tournament by ω-categoricity.

It remains to show that each automorphism of T_0 extends to a unique automorphism of T. The extension property is clear by construction. For unicity, one sees as before Claim 2.2 that an automorphism of T which stabilizes T_0 setwise stabilizes each T_s $(s \geq 0)$ setwise, and is thus uniquely determined by its action on T_0, again by the construction. The proof of Theorem 1.1 is now complete. ∎

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